Short Course in Mathematics and Analytic Geometry
Week 6 Answers

1 Matrix Operations

1.1 Addition

Solutions to the following matrix sums:

1. $\begin{bmatrix} 1 & 2 & 5 \\ 8 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 2 & 5 \\ 8 & 8 & 0 \\ 3 & 5 & 15 \end{bmatrix}$
2. $\begin{bmatrix} x & y & z \end{bmatrix} + \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} x+a & y+b & z+c \end{bmatrix}$
3. $\begin{bmatrix} 3 & 4 \\ 5 & 7 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 8 \\ 2 & 0 \end{bmatrix}$

1.2 Multiplication

Solutions to the following matrix multiplications:

4. $\begin{bmatrix} 1 & 2 & 5 \\ 8 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 8 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix}$

5. $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax & by & cz \end{bmatrix}$

6. $\begin{bmatrix} 3 & 4 \\ 5 & 7 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 1 \\ -2 & 7 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 5 \\ 8 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & 2y & 5z \\ 8x & y & 0 \\ 3x & 5y & 9z \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 5 & 1 \\ 8 & 1 & 0 & -5 \\ 3 & 5 & 9 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 7 \\ 1 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 17 & 20 \\ 34 & 54 \\ 40 & 47 \\ 12 & 9 \end{bmatrix}$

2 Determinants

Determinant solutions:

10. $2 \times 3 \times 4 = 24$

11. $\begin{vmatrix} 7 & 2 & 2 \\ 4 & 2 & 2 \\ 9 & 2 & 2 \end{vmatrix} = 0$

12. $\begin{vmatrix} 1 & 1 & 1 \\ 5 & 8 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$

13. $\begin{vmatrix} 1 & 2 & 5 \\ 8 & 1 & 0 \\ 3 & 5 & 9 \end{vmatrix} = 50$

14. $\begin{vmatrix} 1 & 8 & 3 \\ 2 & 1 & 5 \\ 5 & 0 & 9 \end{vmatrix} = 50$

15. $\begin{vmatrix} 1 & 2 & 5 & 1 & 2 \\ 8 & 1 & 0 & -5 & -5 \\ 3 & 5 & 9 & 3 & 9 \\ 1 & 2 & 3 & 4 & -7 \\ -2 & -1 & -9 & 4 & -3 \end{vmatrix} = -623$

3 Simultaneous Equations

Solutions to simultaneous equation using Cramer's rule:

16. $\begin{vmatrix} 7 & -2 \\ 1 & 5 \end{vmatrix} = 37$

$37x = \begin{vmatrix} 3 & -2 \\ 9 & 5 \end{vmatrix} = 33 \;\therefore\; x = \frac{33}{37}$

$37y = \begin{vmatrix} 7 & 3 \\ 1 & 9 \end{vmatrix} = 60 \;\therefore\; y = \frac{60}{37}$

17. $\begin{vmatrix} 1 & 2 & 5 \\ 8 & 1 & 0 \\ 3 & 5 & 9 \end{vmatrix} = 50$

$50x = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 0 \\ 3 & 5 & 9 \end{vmatrix} = 8 \;\therefore\; x = \frac{4}{25}$

$50y = \begin{vmatrix} 1 & 1 & 5 \\ 8 & 2 & 0 \\ 3 & 3 & 9 \end{vmatrix} = 36 \;\therefore\; y = \frac{18}{25}$

$50z = \begin{vmatrix} 1 & 2 & 1 \\ 8 & 1 & 2 \\ 3 & 5 & 3 \end{vmatrix} = -6 \;\therefore\; z = -\frac{3}{25}$

18. $-623v = \begin{vmatrix} 1 & 2 & 5 & 1 & 2 \\ 1 & 1 & 0 & -5 & -5 \\ 1 & 5 & 9 & 3 & 9 \\ 1 & 2 & 3 & 4 & -7 \\ 1 & -1 & -9 & 4 & -3 \end{vmatrix} = 2345 \;\therefore\; v = -\frac{2345}{623} = -\frac{335}{89}$

$$-623w = \begin{vmatrix} 1 & 1 & 5 & 1 & 2 \\ 8 & 1 & 0 & -5 & -5 \\ 3 & 1 & 9 & 3 & 9 \\ 1 & 1 & 3 & 4 & -7 \\ -2 & 1 & -9 & 4 & -3 \end{vmatrix} = -5593 \ \therefore \ w = \tfrac{5593}{623} = \tfrac{799}{89}$$

$$-623x = \begin{vmatrix} 1 & 2 & 1 & 1 & 2 \\ 8 & 1 & 1 & -5 & -5 \\ 3 & 5 & 1 & 3 & 9 \\ 1 & 2 & 1 & 4 & -7 \\ -2 & -1 & 1 & 4 & -3 \end{vmatrix} = 987 \ \therefore \ x = -\tfrac{987}{623} = -\tfrac{141}{89}$$

$$-623y = \begin{vmatrix} 1 & 2 & 5 & 1 & 2 \\ 8 & 1 & 0 & 1 & -5 \\ 3 & 5 & 9 & 1 & 9 \\ 1 & 2 & 3 & 1 & -7 \\ -2 & -1 & -9 & 1 & -3 \end{vmatrix} = 2233 \ \therefore \ y = -\tfrac{2233}{623} = -\tfrac{319}{89}$$

$$-623z = \begin{vmatrix} 1 & 2 & 5 & 1 & 1 \\ 8 & 1 & 0 & -5 & 1 \\ 3 & 5 & 9 & 3 & 1 \\ 1 & 2 & 3 & 4 & 1 \\ -2 & -1 & -9 & 4 & 1 \end{vmatrix} = 525 \ \therefore \ z = -\tfrac{525}{623} = -\tfrac{75}{89}$$

19. $\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

$$x = \begin{vmatrix} 1 & -\sin\theta & 0 \\ 2 & \cos\theta & 0 \\ 3 & 0 & 1 \end{vmatrix} = \cos\theta + 2\sin\theta$$

$$y = \begin{vmatrix} \cos\theta & 1 & 0 \\ \sin\theta & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 2\cos\theta - \sin\theta$$

$$z = \begin{vmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 2 \\ 0 & 0 & 3 \end{vmatrix} = 3\cos^2\theta + 3\sin^2\theta = 3$$

20. $\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix} = 0$ No solutions for x, y and z

Given the line that passes through points $(-1,4,2)$ and $(0,3,3)$, find the point that intersects the planes:

21. The plane $\mathbf{p}(s,t) = \mathbf{r}_1 + s\mathbf{D}_x + t\mathbf{D}_y$, where $\mathbf{r}_1 = \mathbf{i} - \mathbf{j}$, $\mathbf{D}_x = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{D}_y = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

22. The parametric plane passing through points $(1,2,6)$, $(2,3,-1)$ and $(1,0,1)$.

Equation for the line: The line through $(-1,4,2)$ and $(0,3,3)$: $\mathbf{r}(u) = \mathbf{r}_L + u\mathbf{D}_L$ where: $\mathbf{r}_L = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{D}_L = -\mathbf{i} + \mathbf{j} - \mathbf{k}$.

21: We want to solve $\mathbf{r}(u) = \mathbf{p}(s,t)$:

$$\begin{aligned} \mathbf{r}_L + u\mathbf{D}_L &= \mathbf{r}_1 + s\mathbf{D}_x + t\mathbf{D}_y \\ \mathbf{r}_L - \mathbf{r}_1 &= s\mathbf{D}_x + t\mathbf{D}_y - u\mathbf{D}_L \end{aligned}$$

$$\begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & -2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 0 \;\therefore \text{The line never intersects the plane.}$$

22: Given points $(1,2,6)$, $(2,3,-1)$ and $(1,0,1)$, a position vector on the plane is $\mathbf{r}_0 = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, and two vectors parallel to the plane are: $\mathbf{D}_x = \mathbf{i} + \mathbf{j} - 7\mathbf{k}$ and $\mathbf{D}_* = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. However, $\mathbf{D}_x$ and $\mathbf{D}_*$ are not orthogonal, as $\mathbf{D}_x \cdot \mathbf{D}_* \neq 0$. We can use vector projection to get a $\mathbf{D}_y$ vector that is orthogonal to $\mathbf{D}_x$:

$$\begin{aligned} proj_{\mathbf{D}_x}\mathbf{D}_* &= \frac{\mathbf{D}_x \cdot \mathbf{D}_*}{\parallel \mathbf{D}_x \parallel^2}\mathbf{D}_x \\ proj_{\mathbf{D}_x}\mathbf{D}_* &= \left(\frac{6}{17}\right)\mathbf{D}_x \\ \mathbf{D}_y &= \mathbf{D}_* - \left(\frac{6}{17}\right)\mathbf{D}_x \\ \mathbf{D}_y &= \frac{11}{17}\mathbf{i} + \frac{45}{17}\mathbf{j} + \frac{8}{17}\mathbf{k} \end{aligned}$$

Since, $\mathbf{D}_x \cdot \mathbf{D}_y = 0$, we can define a rectangular parametric plane: $\mathbf{p}(s,t) = \mathbf{r}_0 + s\mathbf{D}_x + t\mathbf{D}_y$, that contains our initial three points (note: we could have also solved the problems with $\mathbf{D}_*$ instead of $\mathbf{D}_y$ as the solution point lies on the plane). Given the line $\mathbf{r}(u)$, we want to solve:

$$\begin{aligned} \mathbf{r}_L + u\mathbf{D}_L &= \mathbf{r}_0 + s\mathbf{D}_x + t\mathbf{D}_y \\ \mathbf{r}_L - \mathbf{r}_0 &= s\mathbf{D}_x + t\mathbf{D}_y - u\mathbf{D}_L \end{aligned}$$

Expressing in matrix form:

$$\begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{11}{17} & 1 \\ 1 & \frac{45}{17} & -1 \\ -7 & \frac{8}{17} & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

Finding the determinant:

$$\begin{vmatrix} 1 & \frac{11}{17} & 1 \\ 1 & \frac{45}{17} & -1 \\ -7 & \frac{8}{17} & 1 \end{vmatrix} = 26$$

We can easily solve for u:

$$26u = \begin{vmatrix} 1 & \frac{11}{17} & -3 \\ 1 & \frac{45}{17} & 1 \\ -7 & \frac{8}{17} & 3 \end{vmatrix} = -56 \ \therefore \ u = -\frac{28}{13}$$

The line $\mathbf{r}(u)$ where $u = -\frac{28}{13}$ gives the position vector: $\frac{15}{13}\mathbf{i} + \frac{24}{13}\mathbf{j} + \frac{54}{13}\mathbf{k}$. The line $\mathbf{r}(u)$ intersects the plane $\mathbf{p}(s,t)$ at point:

$$\left(\frac{15}{13}, \frac{24}{13}, \frac{54}{13}\right)$$

.

Proof:

$$\begin{aligned} \mathbf{D}_x \times \mathbf{D}_* &= 19\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} \\ (\mathbf{r}_L - \mathbf{r}(-\frac{28}{13})) \cdot (\mathbf{D}_x \times \mathbf{D}_*) &= 0 \end{aligned}$$