Short Course in Mathematics and Analytic Geometry
Week 7 Answers

1 Matrix Inverse

Solution inverse matrices:

1. $\begin{bmatrix} \frac{9}{50} & \frac{7}{50} & -\frac{1}{10} \\ -\frac{36}{25} & -\frac{3}{25} & \frac{4}{5} \\ \frac{37}{50} & \frac{1}{50} & -\frac{3}{10} \end{bmatrix}$
2. $\begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$
3. Not defined
4. $\begin{bmatrix} 0 & 0.80 & 0.60 \\ -0.80 & -0.36 & 0.48 \\ -0.60 & 0.48 & -0.64 \end{bmatrix}$

2 Simultaneous Equations

Simultaneous equations solved inverse matrices:

5. $\begin{bmatrix} 7 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{37} & \frac{2}{37} \\ -\frac{1}{37} & \frac{7}{37} \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{33}{37} \\ \frac{60}{37} \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & 5 \\ 8 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{50} & \frac{7}{50} & -\frac{1}{10} \\ -\frac{36}{25} & -\frac{3}{25} & \frac{4}{5} \\ \frac{37}{50} & \frac{1}{50} & -\frac{3}{10} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{25} \\ \frac{18}{25} \\ -\frac{3}{25} \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 5 & 1 & 2 \\ 8 & 1 & 0 & -5 & -5 \\ 3 & 5 & 9 & 3 & 9 \\ 1 & 2 & 3 & 4 & -7 \\ -2 & -1 & -9 & 4 & -3 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \frac{1}{623} \begin{bmatrix} -3117 & -103 & 898 & 608 & -631 \\ 6921 & 387 & -1874 & -1359 & 1518 \\ -1169 & -84 & 315 & 266 & -315 \\ -2949 & -173 & 849 & 622 & -582 \\ -654 & -39 & 213 & 79 & -124 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{335}{89} \\ \frac{799}{89} \\ -\frac{141}{89} \\ -\frac{319}{89} \\ -\frac{75}{89} \end{bmatrix}$$

3 Eigenvectors and Eigenvalues

8. Given the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 8 \\ 1 & 3 & 2 \\ 6 & -3 & 1 \end{bmatrix}$$

Eigenvalues are:

$$\begin{aligned} \lambda_1 &= 7 \\ \lambda_2 &= \frac{-1-\sqrt{39}}{2} \approx -5.32183 \\ \lambda_3 &= \frac{-1+\sqrt{93}}{2} \approx 4.32183 \end{aligned}$$

Eigenvectors as matrix columns:

$$\mathbf{P} = \begin{bmatrix} 10 & -1.10728 & 2.10728 \\ 6 & -0.10728 & 3.10728 \\ 7 & 1 & 1 \end{bmatrix}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 0.09091 & -0.09091 & 0.09091 \\ -0.445441 & 0.13436 & 0.52118 \\ -0.19092 & 0.50206 & -0.15755 \end{bmatrix}$$

Proof:

$$\mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 70 & 5.89272 & 9.10728 \\ 42 & 0.570899 & 13.4291 \\ 49 & -5.32183 & 4.32183 \end{bmatrix} \mathbf{P}^{-1} = \begin{bmatrix} 2 & -1 & 8 \\ 1 & 3 & 2 \\ 6 & -3 & 1 \end{bmatrix}$$

a. $\mathbf{A}^{20} = \mathbf{P}\begin{bmatrix} 7^{20} & 0 & 0 \\ 0 & (-5.32183)^{20} & 0 \\ 0 & 0 & (4.32183)^{20} \end{bmatrix}\mathbf{P}^{-1} = \begin{bmatrix} 7.27001 \times 10^{16} & -7.25824 \times 10^{16} & 7.23451 \times 10^{16} \\ 4.35359 \times 10^{16} & -4.35198 \times 10^{16} & 4.35020 \times 10^{16} \\ 5.06280 \times 10^{16} & -5.07297 \times 10^{16} & 5.09491 \times 10^{16} \end{bmatrix}$

b. $\exp(\mathbf{A}) = \mathbf{P}\begin{bmatrix} \exp(7) & 0 & 0 \\ 0 & \exp(-5.32183) & 0 \\ 0 & 0 & \exp(4.32183) \end{bmatrix}\mathbf{P}^{-1} = \begin{bmatrix} 966.636 & -917.255 & 971.928 \\ 553.477 & -480.665 & 561.288 \\ 683.474 & -660.043 & 685.993 \end{bmatrix}$

c. To solve this we need the relationship: $\log(-r) = \log(r) + i\pi$, so that:

$$\log(\mathbf{A}) = \mathbf{P}\begin{bmatrix} \log(7) & 0 & 0 \\ 0 & \log(-5.32183) & 0 \\ 0 & 0 & \log(4.32183) \end{bmatrix}\mathbf{P}^{-1}$$

$$= \begin{bmatrix} 2.00473 + 1.54953i & -0.46936 - 0.46738i & 0.31828 - 1.813i \\ 0.27298 + 0.15012i & 1.19764 - 0.04528i & 0.2514 - 0.17565i \\ 0.21416 - 1.3994i & -0.27891 + 0.4221i & 1.87903 + 1.63735i \end{bmatrix}, \text{ otherwise Not Defined in } \mathbb{R}$$

d. $\sin(\mathbf{A}) = \mathbf{P}\begin{bmatrix} \sin(7) & 0 & 0 \\ 0 & \sin(-5.32183) & 0 \\ 0 & 0 & \sin(4.32183) \end{bmatrix}\mathbf{P}^{-1} = \begin{bmatrix} 1.37372 & -1.69745 & 0.431056 \\ 0.946104 & -1.81258 & 0.765191 \\ 0.229374 & -0.772114 & 0.991122 \end{bmatrix}$

e. $\cos(\mathbf{A}) = \mathbf{P}\begin{bmatrix} \cos(7) & 0 & 0 \\ 0 & \cos(-5.32183) & 0 \\ 0 & 0 & \cos(4.32183) \end{bmatrix}\mathbf{P}^{-1} = \begin{bmatrix} 1.12086 & -1.17326 & 0.481428 \\ 0.664424 & -1.01333 & 0.56559 \\ 0.297466 & -0.593967 & 0.838065 \end{bmatrix}$

f. $\exp(\mathbf{A}^2) + \sin(\mathbf{A})$

$$= \mathbf{P}\begin{bmatrix} \exp(7^2) + \sin(7) & 0 & 0 \\ 0 & \exp((-5.32183)^2) + \sin(-5.32183) & 0 \\ 0 & 0 & \exp(((4.32183)^2) + \sin(4.32183) \end{bmatrix}\mathbf{P}^{-1}$$

$$= \begin{bmatrix} 1.7339514303479187 \times 10^{21} & -1.7339514297842658 \times 10^{21} & 1.7339514285967669 \times 10^{21} \\ 1.0403708578027027 \times 10^{21} & -1.0403708577479779 \times 10^{21} & 1.0403708576330597 \times 10^{21} \\ 1.2137659999498802 \times 10^{21} & -1.2137660004588082 \times 10^{21} & 1.2137660015313882 \times 10^{21} \end{bmatrix}$$